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On high-order hypoplastic models for granular materials†

WEI WU
Institute of Geotechnical Engineering, Universität für Bodenkultur, 1180 Vienna, Austria (E-mail: wei.wu@boku.ac.at)

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Abstract. Granular materials show a wealth of interesting phenomena such as fluid-like behaviour and scale dependence. The description of these phenomena lies beyond the capability of conventional constitutive models. The paper discusses some high-order models within the framework of hypoplasticity. The high-order models are formulated by including the temporal and spatial derivatives of strain rate into the constitutive equation.

Key words: granular material, rate dependence, strain gradient

1. Introduction

Granular materials consist of large numbers of discrete particles. Depending on the external agency, granular materials can exhibit solid-like behaviour at rest and fluid-like behaviour during flow. While dense sand at rest is well suited for building foundations, the flow properties play an important role in processing and handling granular materials. Traditionally, the theories for the solid and fluid regimes have been developed independently. Due to their complexity, many aspects of the behaviour of granular materials are not yet well understood. In particular, little is known of the transition between the solid and fluid regimes.

Despite their discrete nature, granular materials can often be reasonably described as continua. If, however, the domain size is of the order of the particle size, the corresponding boundary-value problems will show scale dependence to some extent, *e.g.* the shear-band formation and the interface between granular material and the external boundary. The conventional continuum theories cannot describe the scale dependence, since they do not have a length scale. One possible approach to account for the scale dependence is to enrich the conventional continuum theories with some length scales.

Central to the continuum theories for granular materials is the development of constitutive models. The most widely used continuum models for granular materials are plasticity theory for the solid regime and non-Newtonian fluids for the fluid regime [1]. Recently, hypoplasticity based on nonlinear tensorial functions has been proposed as an alternative to plasticity for the solid-like behaviour of granular materials [2–5]. An exhaustive review of the development of hypoplasticity can be found in the recent work by Wu and Kolymbas [6] and Tamagnini *et al.* [7]. The distinctive features of hypoplasticity are its simple formulation and its capacity to capture some salient features of granular materials, such as nonlinearity, dilatancy and yielding. Since the basic hypoplastic model is for the solid-like behaviour and does not possess an internal length scale, the primary objective of this paper is twofold, *i.e.*, to extend the basic model to a unified description of the solid-like and fluid-like behaviour and to enrich the basic model with an internal length to account for scale dependence.

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2. Background of hypoplasticity

We recapitulate the main ingredients of hypoplasticity and begin with a fairly general formulation by assuming that there exists a tensor-valued function $\mathbf{H}$ such that

$$\mathring{\mathbf{T}} = \mathbf{H}(\mathbf{T}, \mathbf{D}), \tag{1}$$

where $\mathbf{T}$ and $\mathbf{D}$ stand for the stress tensor and the stretching tensor, respectively. The Jaumann stress rate $\mathring{\mathbf{T}}$ in the above is defined by

$$\mathring{\mathbf{T}} = \dot{\mathbf{T}} + \mathbf{TW} - \mathbf{WT}, \tag{2}$$

where $\mathbf{W}$ denotes the spin tensor. The stretching and spin tensors are related to the velocity-gradient tensor through

$$\mathbf{D} = \frac{1}{2}(\partial\mathbf{v}/\partial\mathbf{x} + (\partial\mathbf{v}/\partial\mathbf{x})^T), \qquad \mathbf{W} = \frac{1}{2}(\partial\mathbf{v}/\partial\mathbf{x} - (\partial\mathbf{v}/\partial\mathbf{x})^T). \tag{3}$$

Unless stated otherwise, we use bold-faced letters to denote vectors and tensors. A superposed dot stands for material time differentiation.

Within the framework of (1) we proceed to consider the following *ansatz* for hypoplastic constitutive equations by Wu and Kolymbas [8]

$$\mathring{\mathbf{T}} = \mathbf{L}(\mathbf{T}, \mathbf{D}) + \mathbf{N}(\mathbf{T})\|\mathbf{D}\|, \tag{4}$$

where $\mathbf{L}(\mathbf{T}, \mathbf{D})$ and $\mathbf{N}(\mathbf{T})$ are isotropic tensor-valued functions of their arguments. Here $\mathbf{L}(\mathbf{T}, \mathbf{D})$ is assumed to be linear in $\mathbf{D}$; $\|\cdot\|$ stands for a Euclidean norm of the stretching tensor defined by $\|\mathbf{D}\| = \sqrt{\operatorname{tr}\mathbf{D}^2}$, where tr represents the trace of a tensor. Because of $\sqrt{\operatorname{tr}\mathbf{D}^2}$, the constitutive equation is incrementally nonlinear in $\mathbf{D}$. The *ansatz* (4) defines a class of incrementally nonlinear constitutive equations.

A concrete constitutive equation can be obtained by placing requirements on the *ansatz*. The following requirements are found to be useful in developing the constitutive equations.

Let us confine ourselves to the case of rate independence, which means that the stress rate remains unchanged under a change of time scale.

Requirement 1 For rate independence the function $\mathbf{H}(\mathbf{T}, \mathbf{D})$ in (1) must be positively homogeneous of the first degree in $\mathbf{D}$

$$\mathbf{H}(\mathbf{T}, \lambda\mathbf{D}) = \lambda\mathbf{H}(\mathbf{T}, \mathbf{D}). \tag{5}$$

The second requirement states the objectivity of the constitutive equations under rigid rotations. The objectivity requirement is fulfilled if the function $\mathbf{H}(\mathbf{T}, \mathbf{D})$ in (1) is isotropic ([9]).

Requirement 2 The function $\mathbf{H}(\mathbf{T}, \mathbf{D})$ must fulfil the following condition of objectivity

$$\mathbf{H}(\mathbf{QTQ}^{\mathrm{T}}, \mathbf{QDQ}^{\mathrm{T}}) = \mathbf{QH}(\mathbf{T}, \mathbf{D})\mathbf{Q}^{\mathrm{T}}, \tag{6}$$

where $\mathbf{Q}$ is an orthogonal tensor.

The requirement of objectivity is satisfied if the function $\mathbf{H}(\mathbf{T}, \mathbf{D})$ is formulated according to the representation theorems for isotropic tensor-valued functions. In the most general case, the representation theorem for a tensor-valued function of two symmetric tensors can be written out as follows ([10])

$$\begin{aligned}\mathring{\mathbf{T}} = \phi_0\mathbf{1} + \phi_1\mathbf{T} + \phi_2\mathbf{D} + \phi_3\mathbf{T}^2 + \phi_4\mathbf{D}^2 + \phi_5(\mathbf{TD} + \mathbf{DT}) \\ + \phi_6(\mathbf{T}^2\mathbf{D} + \mathbf{DT}^2) + \phi_7(\mathbf{TD}^2 + \mathbf{D}^2\mathbf{T}) + \phi_8(\mathbf{T}^2\mathbf{D}^2 + \mathbf{D}^2\mathbf{T}^2),\end{aligned} \tag{7}$$

where $\mathbf{1}$ is the unit tensor. The coefficients $\phi_i\,(i = 0\ldots 8)$ are functions of the invariants and joint invariants of $\mathbf{T}$ and $\mathbf{D}$:

$$\phi_i = \phi_i(\mathrm{tr}\,\mathbf{T}, \mathrm{tr}\,\mathbf{T}^2, \mathrm{tr}\,\mathbf{T}^3, \mathrm{tr}\,\mathbf{D}, \mathrm{tr}\,\mathbf{D}^2, \mathrm{tr}\,\mathbf{D}^3, \mathrm{tr}\,(\mathbf{TD}), \mathrm{tr}\,(\mathbf{T}^2\mathbf{D}), \mathrm{tr}\,(\mathbf{TD}^2), \mathrm{tr}\,(\mathbf{T}^2\mathbf{D}^2)). \tag{8}$$

The third requirement is based on the following experimental observation on dry sand by Goldscheider ([11]): *A proportional strain (stress) path starting from a nearly stress-free and undistorted state gives rise to a proportional stress (strain) path.* In experiment, an initial stress is often required to hold a specimen of cohesionless granular material together. Frequently, this initial stress does not lie on the proportional stress path. For this case, Chu and Lo [12] showed that the stress state will approach a proportional stress path asymptotically. The proportionality between stress and strain was also investigated by Hill [13]. Hill formulated homogeneous stress-strain relations based on the assumption that the components of strain (or strain-rate as the case may be) increase in fixed ratio when the components of stress do. The proportionality between stress and strain has some important bearing on the constitutive model, which can be formulated as follows.

Requirement 3 The function $\mathbf{H}(\mathbf{T}, \mathbf{D})$ should be homogeneous in $\mathbf{T}$, *i.e.*

$$\mathbf{H}(\lambda\mathbf{T}, \mathbf{D}) = \lambda^n\mathbf{H}(\mathbf{T}, \mathbf{D}), \tag{9}$$

where λ is an arbitrary scalar and n denotes the degree of homogeneity. The stiffness and strength is found to be proportional to the n-th power of the stress level, which is represented by the trace of the stress tensor $\mathrm{tr}\,\mathbf{T}$. For $n = 1$ the stiffness and strength depend linearly on the stress level. A typical example of $n = 1$ is the Mohr–Coulomb strength criterion. Deviation from the linear dependence is often observed at extemely low and high stress levels. In developing constitutive models, it is advisable to start with the simplest case of $n = 1$.

An important property of granular materials is the so-called yielding. At yielding the stress rate approaches null and the shear strength is exhausted. The yielding condition for the general constitutive equation is defined as follows

Definition A material element is said to be yielding if, for a given stress state $\mathbf{T}$, there exists a $\mathbf{D}$ such that

$$\mathring{\mathbf{T}} = \mathbf{H}(\mathbf{T}, \mathbf{D}) = \mathbf{0}. \tag{10}$$

If the total set of $\mathbf{T} \in \mathcal{L} = \{\mathbf{T} \mid \mathring{\mathbf{T}} = \mathbf{0}\}$ forms a surface in the stress space, it will be called the yield surface. Note that any $\mathbf{T} \in \mathcal{L}$ is accompanied by the stretching $\mathbf{D} \in \mathcal{K} = \{\mathbf{D} \mid \mathring{\mathbf{T}} = \mathbf{0}\}$. Yielding is characterized by the pair $(\mathbf{T}, \mathbf{D})$. On the yield surface there exists at least one $\mathbf{D}$ such that the stress rate vanishes.

As an example, the following specific constitutive equation by Wu and Bauer [14] has been used to solve some boundary-value problems [15,16]

$$\mathring{\mathbf{T}} = C_1(\mathrm{tr}\,\mathbf{T})\mathbf{D} + C_2\frac{\mathrm{tr}(\mathbf{TD})\mathbf{T}}{\mathrm{tr}\,\mathbf{T}} + \left(C_3\frac{\mathbf{T}^2}{\mathrm{tr}\,\mathbf{T}} + C_4\frac{\mathbf{T}^{*2}}{\mathrm{tr}\,\mathbf{T}}\right)\|\mathbf{D}\|, \tag{11}$$

where C_i $(i=1,\ldots,4)$ are dimensionless constants. The deviatoric stress tensor $\mathbf{T}^*$ in (11) is defined by $\mathbf{T}^*=\mathbf{T}-1/3(\mathrm{tr}\,\mathbf{T})\mathbf{1}$. It can be easily ascertained that constitutive equation (11) fulfils the Requirements 1 to 3.

3. Rate-dependent model

The hypoplastic constitutive model in the last section is meant primarily to describe the solid-like or statical behaviour of granular materials, which is the main concern in geotechnical engineering. However, the same granular materials may exhibit fluid-like or dynamical behaviour, which is of importance in numerous engineering problems, *e.g.* bulk solids handling, debris flow, bed-load transport and fluidization [17, 18]. The dominating mechanism in the solid-like regime is frictional, while the fluid-like behaviour is primarily viscous. Moreover, the viscous behaviour of granular materials differs from that of Newtonian fluids in several aspects, *e.g.* the dependence of viscosity on the volume fraction (void ratio) of granular materials. We will confine ourselves to dense granular materials or concentrated suspensions and consider granular materials as a single-phase medium.

There is an extensive experimental data base in the literature on the flow properties of granular materials. One of the most influential pieces of work was due to Bagnold [19]. Bagnold's experiments were carried out on suspensions of neutrally buoyant particles in a Couette-flow apparatus. Based on his experimental results, Bagnold distinguished three flow regimes, namely a macro-viscous regime (low shear rate), a grain-inertia regime (high shear rate) and a transitional regime. The dependence of the shear and normal stress on the shear rate is found to be linear at low shear rate and quadratic at high shear rate. More recent experimental work includes that of Savage and McKeown [20] and also Hanes and Inman [21]. Some salient features of dense granular flow can be summarised as follows:

– dependence of viscosity on the particle concentration;
– normal stress effect and dilatancy;
– different behaviour for loading and unloading;
– localized flow pattern (shear bands).

The normal-stress effect, known as the Weissenberg effect, can be demonstrated in a Couette flow between two rotating coaxial cylinders. The fluid surface near the rotating cylinder is found to climb. The climbing of the fluid surface in non-Newtonian fluids is ascribed to the normal stress generated during shearing. A detailed exposition on this subject with numerous photos can be found in the monograph by Truesdell and Noll [9].

Traditionally, the description of the fluid-like behaviour has been developed independently of the theories for the solid-like behaviour. One is tempted to treat granular flow as a non-Newtonian fluid with a single viscosity.

$$\mathbf{T}=\eta\,\mathbf{D}, \tag{12}$$

where η is the viscosity coefficient. However, non-Newtonian models within the framework of (12) fail to capture the normal-stress effect. As a consequence, most early models were based on the following expression (see the review by Wang and Hutter [22])

$$\mathbf{T}=\alpha_1\mathbf{1}+\alpha_2\mathbf{D}+\alpha_3\mathbf{D}^2 \tag{13}$$

The coefficients α_i $(i=1\ldots 3)$ depend on $\mathbf{D}$ and the volume fraction. In some recent models the coefficients in (13) are allowed to depend on the gradient of volume fraction [23, 24].

All models proposed until now are based on a relationship between stress and strain rate. Such models cannot account for the different behaviour for loading and unloading, which calls for constitutive models formulated in rate. Moreover, most constitutive models treat the solid-like and fluid-like behaviour separately. A unified model applicable to both the solid-like and fluid-like state is still missing. We will follow a different apparoach by extending the hypoplastic constitutive model given in the last section.

At first we assume that the stress tensor and its rate can be decomposed into two parts, namely a first part for the statical (rate-independent) behaviour and a second part for the dynamical (rate-dependent) behaviour, *viz.*

$$\mathbf{T} = \hat{\mathbf{T}} + \check{\mathbf{T}}, \qquad \mathring{\mathbf{T}} = \mathring{\hat{\mathbf{T}}} + \mathring{\check{\mathbf{T}}}. \tag{14}$$

The statical part is described by the constitutive equation (1), which can be written out using $\hat{\mathbf{T}}$ as follows

$$\mathring{\hat{\mathbf{T}}} = \hat{\mathbf{H}}(\hat{\mathbf{T}}, \mathbf{D}). \tag{15}$$

The dynamical part has to be formulated in rate as well. To this end the equation for Newtonian fluid (12) can be rewritten in the following rate form

$$\mathring{\check{\mathbf{T}}} = \eta\, \mathring{\mathbf{D}}, \tag{16}$$

where $\mathring{\mathbf{D}}$ is the Jaumann stretching-rate tensor and can be obtained according the scheme in (2), that is,

$$\mathring{\mathbf{D}} = \dot{\mathbf{D}} + \mathbf{D}\mathbf{W} - \mathbf{W}\mathbf{D}. \tag{17}$$

The Jaumann stretching rate can be compared to the Rivlin–Ericksen tensor [9]. Equation (16) can be formally regarded as the time differentiation of (12) provided the viscosity coefficient is constant. It is interesting to notice that similar high-order terms have been proposed by Oldroyd [25] for fluids with elasticity and by Kolymbas [26] for the creep of cohesive soil. In general, the viscosity may depend on shear rate, so that the constitutive equation can be expressed by the following isotropic function:

$$\mathring{\check{\mathbf{T}}} = \check{\mathbf{H}}(\mathbf{D}, \mathring{\mathbf{D}}). \tag{18}$$

Note that the inclusion of high temporal derivative of strain rate in (18) requires the specification of an initial value for $\mathbf{D}$. This can be compared to the specification of the initial speed in an accelerated motion. Note further that the function $\check{\mathbf{H}}$ is assumed to be independent of stress, which seems to be confirmed by some experiments [19, 20].

In searching for a concrete constitutive equation the representation theorem given by (7) and (8) is useful. The following terms can be used to compile a workable constitutive equation

$$\mathring{\check{\mathbf{T}}} = \beta_1 \mathring{\mathbf{D}} + \beta_2 (\mathbf{D}\mathring{\mathbf{D}} + \mathring{\mathbf{D}}\mathbf{D}) + \beta_3 \mathring{\mathbf{D}}^2. \tag{19}$$

The last two terms are included to describe the normal-stress effect. In order to see this, let us consider a simple shear motion described by the following expressions

$$x_1 = X_1 + 2X_2\gamma, \quad x_2 = X_2, \quad x_3 = X_3, \tag{20}$$

with $\mathbf{X}$ and $\mathbf{x}$ denoting the initial and the instantaneous configuration, respectively. Here γ stands for the shearing. To make it simple, we confine ourselves to the material time differention and write out the terms $\mathbf{D}$, $\mathbf{D}\dot{\mathbf{D}}+\dot{\mathbf{D}}\mathbf{D}$ and $\dot{\mathbf{D}}^2$ in matrices as follows

$$\dot{\mathbf{D}}=\begin{bmatrix} 0 & \ddot{\gamma} & 0 \\ \ddot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{D}\dot{\mathbf{D}}+\dot{\mathbf{D}}\mathbf{D}=\begin{bmatrix} \dot{\gamma}\ddot{\gamma} & 0 & 0 \\ 0 & \dot{\gamma}\ddot{\gamma} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \dot{\mathbf{D}}^2=\begin{bmatrix} \ddot{\gamma}^2 & 0 & 0 \\ 0 & \ddot{\gamma}^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{21}$$

It is clear that the last two terms give rise to normal stresses in a simple shear motion. The first term alone, which corresponds to a Newtonian fluid, does not account for the normal-stress effect. It can be shown that the co-rotational part of the Jaumann-stretching-rate tensor in (17) yields normal stresses as well. However, the normal stresses from the Jaumann rate cannot be adjusted to fit experimental data.

Again, the coefficients β_i $(i=1\ldots3)$ in (19) depend on the invariants and joint invariants of $\mathbf{D}$ and $\mathring{\mathbf{D}}$. Let us consider possible forms of the coefficients and take the first term in the above expression as an example. We proceed to consider the following expression

$$\check{\mathring{\mathbf{T}}}=\eta_1\sqrt{\eta_2^2+\operatorname{tr}\mathbf{D}^2}\mathring{\mathbf{D}}, \tag{22}$$

where η_1 and η_2 are parameters to be determined by fitting experimental data, *e.g.* those by Hanes and Inman [21]. The above expression describes fairly well the dependence of viscosity on shear rate in granular flow, namely linear dependence at low shear rate and quadratic dependence at high shear rate.

Combining (15) and (18) gives rise to the complete constitutive equation for both solid-like and fluid-like behaviour:

$$\mathring{\mathbf{T}}=\hat{\mathbf{H}}(\hat{\mathbf{T}},\mathbf{D})+\check{\mathbf{H}}(\mathbf{D},\mathring{\mathbf{D}}). \tag{23}$$

The above constitutive equation applies to the entire process from solid-like to fluid-like behaviour. Unlike most conventional models, where constitutive equations for the statical and dynamical regimes are formulated and applied separately, the above constitutive equation makes no distinction between them. Rather the transition from solid-like to fluid-like behaviour turns out as an outcome.

Let us consider two special cases, where the first term on the r.h.s. of (23) vanishes. From Requirement 3 we know that the constitutive model (4) is homogeneous in stress. This can be ascertained by the specific version (11). For $\hat{\mathbf{T}}=\mathbf{0}$ the first term vanishes identically, irrespective of the strain rate. This situation is expected in the liquefaction of sandy soil during earthquakes. The second case, where the first term vanishes, is met at yielding. Note that the yielding as defined by (10) is for specific strain rates and in general not stationary. If the strain rate deviates from the strain rate at yielding, the stress rate from the first term will not be zero. It is therefore advisable to work with the complete constitutive model, which is supported by the large stress fluctuations in the experiments.

As might be expected, the parameters in (23) depend on the void ratio (suspension concentration). For the statical part the critical void ratio plays an important role. The reader is referred to [27] for an extended hypoplastic model with critical state. For the dynamical part a similar critical density, known as the densest concentration, can also be defined [28]. It is interesting to know whether the void ratio in the statical and dynamical regimes is amenable to a unified description.

In most experiments, the dynamical stress during granular flow is very low. Therefore, for most geotechnical problems the dynamical stress can be neglected. However, there are numerous problems with extremely low statical stress, *e.g.* surface flow and fluidization, where the dynamical stress has to be considered. As an example, let us consider a water-saturated sand layer under simple shear motion. The sand layer is initially under a given effective stress. Provided no drainage is allowed, the shear motion will be isochoric. Along with increasing shear deformation the statical stress decreases. Initially the contribution of the dynamical stress is negligibly small. With decreasing statical stress, however, the motion will be accelerated. This in turn gives rise to a further decrease of the statical stress and increase of the dynamical stress. This process evolves to a critical point, where the statical stress approaches zero and the dynamical stress becomes dominant. This *gedanke*-experiment plays an important role in the post-liquefaction analysis of sandy soil during strong earthquakes.

4. Gradient model

The constitutive models presented until now do not have an internal length scale. Such constitutive models obey the principle of local actions and are called local models in continuum mechanics. Shear-band analyses with local models provide only information about the bifurcation point and the inclination of shear band. The thickness of shear bands and the spacing between them cannot be obtained. In the post-bifurcation regime the numerical analyses with local models depend on the mesh size. Frequently the thickness of shear bands is of the order of several particle diameters. The validity of such constitutive models is limited by the discrete nature of granular materials.

Recently, several continuum models with an internal length scale have been proposed. Most models are developed by enriching the theory of plasticity with higher-order terms. These models can be classified into two groups. The first group presents a radical modification of the local model by including additional statical and kinematical variables, constitutive equations, equilibrium equations and boundary conditions. The representatives of this group are the Cosserat theory [29, 30] and the strain-gradient theory [31, 32]. The second group presents a moderate modification of the local model by including only the strain gradient [33–35]. In general, the first approach gives rise to more material parameters and requires more information from experiments than the second approach. Frequently the only available observation from experiments is the thickness of shear band. In order to determine the additional parameters, the constitutive model should be made as simple as possible.

We will follow the second approach and modify constitutive equation (4) by including higher-order terms. The major questions are what gradient term should be used and how the gradient term may find its way into the constitutive equation. In gradient plasticity theory, the yield function [33] and dilatancy function [34] are assumed to depend on the plastic strain and its gradient. In hypoplasticity, however, the concepts in plasticity theory, such as yield function, flow rule and the decomposition of deformation into elastic and plastic parts, are not used. Equation (4) does not contain strain but strain rate. Therefore we should use the strain-rate gradient instead of the strain gradient.

According to the principle of local actions, the deformation at one point affects only the material behaviour in an infinitely small vicinity [9]. This remains valid as long as the deformation is homogeneous. For heterogeneous deformation the interaction of material in a finite volume has to be considered. In general the response depends on the averaged deformation over the finite volume. However, the average of deformation need not be applied to the whole constitutive equation. The nonlocal theories of plasticity are often obtained by assuming local elasticity and nonlocal plasticity.

To this end, constitutive equation (4) can be regarded as composed of two parts, namely a linear part, which depends on the stretching tensor through a linear tensorial function, and a nonlinear part, which depends on the strain rate through its norm. We proceed to modify Equation (4) by assuming that the nonlinear part depends on the averaged stretching norm for heterogeneous deformation.

The averaged stretching norm can be obtained by looking at the Taylor expansion of $\|\mathbf{D}\|$ in the vicinity of $\mathbf{x}$

$$\|\mathbf{D}\|(\mathbf{x}+\Delta\mathbf{x}) = \|\mathbf{D}\|(\mathbf{x}) + \nabla\|\mathbf{D}\| : \Delta\mathbf{x} + \frac{1}{2!}\nabla(\nabla\|\mathbf{D}\|) : (\Delta\mathbf{x}\otimes\Delta\mathbf{x}) + \cdots, \tag{24}$$

where ∇ is the Nabla operator and $\otimes$ denotes a tensor product. The above expression can be integrated over a sphere with the radius $R = \|\Delta\mathbf{x}\|$ to give the averaged stretching norm. The following expression can be obtained by retaining terms up to the second order

$$\widehat{\|\mathbf{D}\|} \approx \|\mathbf{D}\| + \frac{1}{V_s}\frac{2\pi R^5}{15}\nabla^2\|\mathbf{D}\|, \tag{25}$$

where $\widehat{\|\mathbf{D}\|}$ is the averaged stretching norm and V_s is the volume of the sphere. Inserting $V_s = (4/3)\pi R^3$ into the above expression leads to

$$\widehat{\|\mathbf{D}\|} \approx \|\mathbf{D}\| + \frac{R^2}{10}\nabla^2\|\mathbf{D}\|. \tag{26}$$

Replacing the local stretching norm in (4) by the averaged stretching norm and denoting $\lambda^2 = R^2/10$, we arrive at the following constitutive equation

$$\mathring{\mathbf{T}} = \mathbf{L}(\mathbf{T},\mathbf{D}) + \mathbf{N}(\mathbf{T})\left[\|\mathbf{D}\| + \lambda^2\nabla^2\|\mathbf{D}\|\right]. \tag{27}$$

The material parameter λ has the length dimension and is assumed to depend on the mean grain diameter. The nonlocal model is a simple modification of the local model (4) and contains only one additional parameter, which can be identified by investigating the thickness of shear band in a plane-strain test. Note that the Laplacian of the stretching norm in the above equation is rather lengthy for three dimensional stress and strain. In general, boundary-value problems with heterogeneous deformation have to be solved numerically, *e.g.* the benchmark problem of plane simple shear with finite thickness and infinite extensions. For homogeneous deformation the Laplacian of the stretching norm is negligibly small and the local model is recovered.

It is worth noting that a nonlocal hypoplastic model based on the integration of stretching over a representative volume has been proposed by Maier [36] and Tejchman [37]. The internal length is obtained by introducing a weight function for the integration. Note that the gradient model in this paper makes use of the differential of stretching, whereas the nonlocal model by Maier and Tejchaman is based on the integral. Therefore, there will be major differences in the numerical implementation of the models.

In order to compare the local and nonlocal models in some detail, let us consider the stress and stretching at yielding. According to the definition (10), the stress rate vanishes at yielding. Constitutive equation (27) can be manipulated to give the following flow rule at yielding

$$\frac{\mathbf{D}}{\|\mathbf{D}\|} = -\mathsf{L}^{-1} : \mathbf{N}\left[1 + \lambda^2\frac{\nabla^2\|\mathbf{D}\|}{\|\mathbf{D}\|}\right], \tag{28}$$

where $\mathsf{L} = \partial\mathbf{L}(\mathbf{T},\mathbf{D})/\partial\mathbf{D}$. The colon denotes an inner product of two tensors. The flow rule given by the above expression is interesting in that the gradient term effects only the magnitude of stretching but not its direction.

From gradient plasticity we know that the gradient term increases the material strength and regularises the underlying boundary problem to some extent. From the flow rule at yielding (28) the yielding function can be readily derived by making use of the fact $\mathbf{D}/\|\mathbf{D}\| : \mathbf{D}/\|\mathbf{D}\| = 1$

$$f(\mathbf{T}, \nabla^2\|\mathbf{D}\|/\|\mathbf{D}\|) = \mathbf{N}^{\mathrm{T}} : \left(\mathsf{L}^{\mathrm{T}}\right)^{-1} : \mathbf{N} : \mathsf{L}^{-1} : \mathbf{N}\left[1 + \lambda^2 \frac{\nabla^2\|\mathbf{D}\|}{\|\mathbf{D}\|}\right]^2 - 1 = 0, \tag{29}$$

where $f(\mathbf{T}, \nabla^2\|\mathbf{D}\|/\|\mathbf{D}\|)$ denotes the yielding function. The yielding function can be evaluated as an average over a representative volume to give a yielding function of stress alone.

The effect of the gradient term on the yielding function can be explained as follows. At yielding the linear part and the nonlinear part in constitutive equation (4) hold in balance. A change of either part leads inevitably to a change of the overall response. The gradient term in the brackets of the above expression can be thought of as a perturbation with $\nabla^2\|\mathbf{D}\|/\|\mathbf{D}\| \ll 1$. The gradient term reduces the nonlinear part and gives rise to an increase in strength.

From a mathematical point of view, the gradient terms are required to regularise the boundary/initial-value problems, which become otherwise ill-posed in the post-bifurcation regime. For the initial-value problems to be well-posed, the underlying PDE should be hyperbolic [38]. The well-posedness of initial-value problems is dictated by the corresponding eigenvalue problem, which can be obtained either by applying Fourier transforms (spatially periodic solutions) to the system of PDE [39] or by considering the propagation of weak discontinuities [40–42]. However, most investigations in the literature are concerned with incrementally linear constitutive models. Little is known about well-posedness with incrementally nonlinear constitutive models. The recent investigation on shear-band formation in hypoplasticity indicates that incrementally nonlinear models are more prone to instability than incrementally linear models [43–45]. For a post-bifurcation analysis a regularisation via gradient terms is required. Since the type of PDE is determined by the high-order terms, the gradient term in (27) is expected to affect the eigenvalue problem and the well-posedness. The relevant research is in progress and will be reported later.

Our last remark concerns the boundary conditions in the gradient models. In principle, higher-order terms in constitutive equations require additional boundary conditions. For constitutive equations involving the Laplacian of strain rate, the gradient of the strain rate on the boundary need to be specified [46, 47]. However, such boundary conditions may be mathematically correct but not necessarily physically sound, since they can hardly be realised in the laboratory. An alternative is the so-called lower-order gradient theory [35, 48], where the gradient term does not appear explicitly in the constitutive equation. In Bassani's gradient model [35], only the tangential stiffness is assumed to depend on the strain gradient. For lower-order gradient theories it is not necessary to specify additional boundary conditions.

5. Conclusions

In this paper some ideas within the framework of hypoplasticity have been presented to describe the fluid-like behaviour and scale dependence of granular materials. The rate-dependent model and the gradient model are developed by including high-order terms, namely the temporal and spatial derivative of the strain rate. In both cases rather simple formulations with few material parameters have been obtained. The models presented in this paper have high potential for some problems, which are beyond the scope of conventional continuum theories, *e.g.* scale dependence and phase transition between solid-like and fluid-like behaviour.

On the way to application, however, two obstacles must be removed. First, the parameters in the models cannot be determined by conventional experiments, where the stresses are uniform and the strains homogeneous. This difficulty can be circumvented by studying some simple boundary-value problems, *e.g.* the shearing of a layer of granular material [49] and the flow of granular material down an incline [50]. The former problem can serve as a benchmark to determine the parameter in the high-order model, while the latter problem can be used for the parameter identification of the rate-dependent model. Second, conventional numerical methods like the finite-element and finite-difference methods are not well suited for our models. In particular, the transition between the solid-like and fluid-like behaviour is often accompanied by large distortions, which may cause severe difficulties for the conventional numerical methods. A promising alternative is the Particle-in-Cell method, which combines the advantages of the Eulerian and Lagrangian formulations [51].

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